

A CLASS OF KÄHLER EINSTEIN STRUCTURES ON THE NONZERO COTANGENT BUNDLE OF A SPACE FORM

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We obtain a class of Kähler Einstein structures on the nonzero cotangent bundle of a Riemannian manifold of positive constant sectional curvature. The obtained class of Kähler Einstein structures depends on one essential parameter, cannot have constant holomorphic sectional curvature and is not locally symmetric.

AMS 2000 Subject Classifications: 53C07, 53C15, 53C55.

Key words: cotangent bundle, Kähler Einstein manifolds.

Abbreviated title: A CLASS OF KÄHLER EINSTEIN STRUCTURES

1.INTRODUCTION

In the study of the differential geometry of the cotangent bundle T^*M of a Riemannian manifold (M, g) one uses several Riemannian and semi-Riemannian metrics, induced from the Riemannian metric g on M . Next, one can get from g some natural almost complex structures on T^*M . The study of the almost Hermitian structures induced from g on T^*M is an interesting problem in the differential geometry of the cotangent bundle.

In [9] the authors have obtained a class of natural Kähler Einstein structures (G, J) of diagonal type induced on T^*M from the Riemannian metric g . The obtained Kähler structures on T^*M depend on two essential parameters a_1 and λ , which are smooth functions depending on the energy density t on T^*M . In the case where the considered Kähler structures are Einstein they get several situations in which the parameters a_1, λ are related by some algebraic relations. In the general case, (T^*M, G, J) has constant holomorphic curvature.

In the present paper we study the singular case where the parameter $a_1 = A\sqrt{t}$, $A \in \mathbf{R}$. The class of the natural almost complex structures

J on the nonzero cotangent bundle T_0^*M that interchange the vertical and horizontal distributions depends on one essential parameter b_1 . This parameter is a smooth real function depending on the energy density t on T_0^*M . From the integrability condition for J it follows that the base manifold M must have constant curvature $c = \frac{A^2}{2}$ and the parameter b_1 must fulfill the condition $b_1 > -\sqrt{\frac{c}{2t}}$.

A class of natural Riemannian metrics G of diagonal type on T_0^*M is defined by four parameters c_1, c_2, d_1, d_2 which are smooth functions of t . From the condition for G to be Hermitian with respect to J we get two sets of proportionality relations, from which we can get the parameters c_1, c_2, d_1, d_2 as functions depending on two new parameters λ and μ .

In the case where the fundamental 2-form ϕ , associated to the class of complex structures (G, J) is closed, one finds that $\mu = \lambda'$.

Thus, we get a class of Kähler structures (G, J) on T_0^*M , depending on one essential parameter λ .

Finally, we find out the conditions under which the obtained class of Kähler structures (G, J) on T_0^*M is Einstein .

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices h, i, j, k, l, r, s being always $\{1, \dots, n\}$ (see [3], [7], [9]). We shall denote by $\Gamma(T_0^*M)$ the module of smooth vector fields on T_0^*M .

2. SOME GEOMETRIC PROPERTIES OF T^*M

Let (M, g) be a smooth n -dimensional Riemannian manifold and denote its cotangent bundle by $\pi : T^*M \longrightarrow M$. Recall that there is a structure of a $2n$ -dimensional smooth manifold on T^*M , induced from the structure of smooth n -dimensional manifold of M . From every local chart $(U, \varphi) = (U, x^1, \dots, x^n)$ on M , it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$, on T^*M , as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first n local coordinates q^1, \dots, q^n are the local coordinates x^1, \dots, x^n of its base point $x = \pi(p)$ in the local chart (U, φ) (in fact we have $q^i = \pi^*x^i = x^i \circ \pi$, $i = 1, \dots, n$). The last n local coordinates p_1, \dots, p_n of $p \in \pi^{-1}(U)$ are the vector space

coordinates of p with respect to the natural basis $(dx_{\pi(p)}^1, \dots, dx_{\pi(p)}^n)$, defined by the local chart (U, φ) , i.e. $p = p_i dx_{\pi(p)}^i$.

An M -tensor field of type (r, s) on T^*M is defined by sets of n^{r+s} components (functions depending on q^i and p_i), with r upper indices and s lower indices, assigned to induced local charts $(\pi^{-1}(U), \Phi)$ on T^*M , such that the local coordinate change rule is that of the local coordinate components of a tensor field of type (r, s) on the base manifold M (see [5] for further details in the case of the tangent bundle). An usual tensor field of type (r, s) on M may be thought of as an M -tensor field of type (r, s) on T^*M . If the considered tensor field on M is covariant only, the corresponding M -tensor field on T^*M may be identified with the induced (pullback by π) tensor field on T^*M .

Some useful M -tensor fields on T^*M may be obtained as follows. Let $u, v : [0, \infty) \rightarrow \mathbf{R}$ be smooth functions and let $\|p\|^2 = g_{\pi(p)}^{-1}(p, p)$ be the square of the norm of the cotangent vector $p \in \pi^{-1}(U)$ (g^{-1} is the tensor field of type $(2, 0)$ having the components $(g^{kl}(x))$ which are the entries of the inverse of the matrix $(g_{ij}(x))$ defined by the components of g in the local chart (U, φ)). The components $u(\|p\|^2)g_{ij}(\pi(p))$, p_i , $v(\|p\|^2)p_i p_j$ define M -tensor fields of types $(0, 2)$, $(0, 1)$, $(0, 2)$ on T^*M , respectively. Similarly, the components $u(\|p\|^2)g^{kl}(\pi(p))$, $g^{0i} = p_h g^{hi}$, $v(\|p\|^2)g^{0k}g^{0l}$ define M -tensor fields of type $(2, 0)$, $(1, 0)$, $(2, 0)$ on T^*M , respectively. Of course, all the components considered above are in the induced local chart $(\pi^{-1}(U), \Phi)$.

The Levi Civita connection $\dot{\nabla}$ of g defines a direct sum decomposition

$$(1) \quad TT^*M = VT^*M \oplus HT^*M.$$

of the tangent bundle to T^*M into vertical distributions $VT^*M = \text{Ker } \pi_*$ and the horizontal distribution HT^*M .

If $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \dots, q^n, p_1, \dots, p_n)$ is a local chart on T^*M , induced from the local chart $(U, \varphi) = (U, x^1, \dots, x^n)$, the local vector fields $\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}$ on $\pi^{-1}(U)$ define a local frame for VT^*M over $\pi^{-1}(U)$ and the local vector fields $\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}$ define a local frame for HT^*M over $\pi^{-1}(U)$, where

$$\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma_{ih}^0 \frac{\partial}{\partial p_h}, \quad \Gamma_{ih}^0 = p_k \Gamma_{ih}^k$$

and $\Gamma_{ih}^k(\pi(p))$ are the Christoffel symbols of g .

The set of vector fields $(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n})$ defines a local frame on T^*M , adapted to the direct sum decomposition (1).

We consider

$$(2) \quad t = \frac{1}{2} \|p\|^2 = \frac{1}{2} g_{\pi(p)}^{-1}(p, p) = \frac{1}{2} g^{ik}(x) p_i p_k, \quad p \in \pi^{-1}(U)$$

the energy density defined by g in the cotangent vector p . We have $t \in [0, \infty)$ for all $p \in T^*M$.

From now on we shall work in a fixed local chart (U, φ) on M and in the induced local chart $(\pi^{-1}(U), \Phi)$ on T^*M .

Now we shall present the following auxiliary result.

LEMMA 1. *If $n > 1$ and u, v are smooth functions on T^*M such that*

$$u g_{ij} + v p_i p_j = 0, \quad p \in \pi^{-1}(U)$$

*on the domain of any induced local chart on T^*M , then $u = 0$, $v = 0$.*

The proof is obtained easily by transvecting the given relation with the components g^{ij} of the tensor field g^{-1} and g^{0j} .

Remark. From the relations of the type

$$u g^{ij} + v g^{0i} g^{0j} = 0, \quad p \in \pi^{-1}(U),$$

$$u \delta_j^i + v g^{0i} p_j = 0, \quad p \in \pi^{-1}(U),$$

it is obtained, in a similar way, $u = v = 0$.

3. A CLASS OF NATURAL COMPLEX STRUCTURES OF DIAGONAL TYPE ON T_0^*M

The nonzero cotangent bundle T_0^*M of Riemannian manifold (M, g) is defined by the formula: T^*M minus zero section. Consider the real valued smooth functions a_1, a_2, b_1, b_2 defined on $(0, \infty)$. We define a class of natural almost complex structures J of diagonal type on T_0^*M , expressed in the adapted local frame by

$$(3) \quad J \frac{\delta}{\delta q^i} = J_{ij}^{(1)}(p) \frac{\partial}{\partial p_j}, \quad J \frac{\partial}{\partial p_i} = -J_{(2)}^{ij}(p) \frac{\delta}{\delta q^j}.$$

where,

$$(4) \quad \begin{cases} J_{ij}^{(1)}(p) = a_1(t)g_{ij} + b_1(t)p_ip_j, \\ J_{(2)}^{ij}(p) = a_2(t)g^{ij} + b_2(t)g^{0i}g^{0j}, \end{cases} \quad A \in \mathbf{R}^*.$$

In this paper we study the singular case where

$$(5) \quad a_1(t) = A\sqrt{t}, \quad A \in \mathbf{R}^*.$$

The components $J_{ij}^{(1)}, J_{(2)}^{ij}$ define symmetric M -tensor fields of types $(0, 2), (2, 0)$ on T^*M , respectively.

PROPOSITION 2. *The operator J defines an almost complex structure on T^*M if and only if*

$$(6) \quad a_1a_2 = 1, \quad (a_1 + 2tb_1)(a_2 + 2tb_2) = 1.$$

Proof. The relations are obtained easily from the property $J^2 = -I$ of J and Lemma 1.

From the relations (5), (6) we can obtain the explicit expression of the parameter a_2, b_2

$$(7) \quad a_2 = \frac{1}{A\sqrt{t}}, \quad b_2 = \frac{-b_1}{At(A + 2\sqrt{t}b_1)}.$$

The obtained class of almost complex structures defined by the tensor field J on T_0^*M is called *class of natural almost complex structures of diagonal type*, obtained from the Riemannian metric g , by using the parameter b_1 . We use the word diagonal for these almost complex structures, since the $2n \times 2n$ -matrix associated to J , with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$ has two $n \times n$ -blocks on the second diagonal

$$J = \begin{pmatrix} 0 & -J_{(2)}^{ij} \\ J_{ij}^{(1)} & 0 \end{pmatrix}.$$

Remark. From the conditions (6) it follows that $a_1 = A\sqrt{t}$ and $a_2 = \frac{1}{A\sqrt{t}}$ cannot vanish and have the same sign. We assume that

$$(8) \quad A > 0.$$

Similarly, from the conditions (6) it follows that $a_1 + 2tb_1$ and $a_2 + 2tb_2$ cannot vanish and have the same sign. We assume that $a_1 + 2tb_1 > 0$, $a_2 + 2tb_2 > 0 \quad \forall t > 0$, i.e.

$$(9) \quad A + 2\sqrt{t}b_1 > 0 \quad \forall t > 0.$$

Now we shall study the integrability of the class of natural almost complex structures defined by J on T_0^*M . To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^i}$, $i = 1, \dots, n$

$$(10) \quad \left[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right] = 0, \quad \left[\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right] = \Gamma_{jk}^i \frac{\partial}{\partial p_k}, \quad \left[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right] = R_{kij}^0 \frac{\partial}{\partial p_k},$$

where $R_{kij}^h(\pi(p))$ are the local coordinate components of the curvature tensor field of $\tilde{\nabla}$ on M and $R_{kij}^0(p) = p_h R_{kij}^h$. Of course, the components R_{kij}^h, R_{kij}^0 define M-tensor fields of types $(1,3), (0,3)$ on T_0^*M , respectively.

Recall that the Nijenhuis tensor field N defined by J is given by

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad \forall X, Y \in \Gamma(T_0^*M).$$

Then, we have $\frac{\delta}{\delta q^k}t = 0$, $\frac{\partial}{\partial p_k}t = g^{0k}$. The expressions for the components of N can be obtained by a quite long, straightforward computation, as follows

THEOREM 3. *The Nijenhuis tensor field of the almost complex structure J on T_0^*M is given by*

$$\begin{cases} N\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right) = \left\{ \frac{A^2}{2}(\delta_i^h g_{jk} - \delta_j^h g_{ik}) - R_{kij}^h \right\} p_h \frac{\partial}{\partial p_k}, \\ N\left(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}\right) = J_{(2)}^{kl} J_{(2)}^{jr} \left\{ \frac{A^2}{2}(\delta_i^h g_{rl} - \delta_r^h g_{il}) - R_{lir}^h \right\} p_h \frac{\delta}{\delta q^k}, \\ N\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = J_{(2)}^{ir} J_{(2)}^{jl} \left\{ \frac{A^2}{2}(\delta_l^h g_{rk} - \delta_r^h g_{lk}) - R_{klr}^h \right\} p_h \frac{\partial}{\partial p_k}. \end{cases}$$

THEOREM 4. *The almost complex structure J on T_0^*M is integrable if and only if (M, g) has positive constant sectional curvature c and*

$$(11) \quad A = \sqrt{2c}.$$

The function b_1 must fulfill the condition

$$(12) \quad b_1 > -\sqrt{\frac{c}{2t}}.$$

Proof. From the condition $N = 0$, one obtains

$$\left\{ \frac{A^2}{2} (\delta_i^h g_{jk} - \delta_j^h g_{ik}) - R_{kij}^h \right\} p_h = 0$$

Differentiating with respect to p_l , it follows that the curvature tensor field of $\dot{\nabla}$ has the expression

$$R_{kij}^l = \frac{A^2}{2} (\delta_i^l g_{jk} - \delta_j^l g_{ik}).$$

Thus (M, g) has positive constant sectional curvature $c = \frac{A^2}{2}$. It follows that $A = \sqrt{2c} > 0$.

Conversely, if (M, g) has positive constant sectional curvature c and A is given by (11), one obtains in a straightforward way that $N = 0$.

Using by the relations (9),(11) we obtain the condition (12).

The class of natural complex structures J of diagonal type on T_0^*M depends on one essential parameter b_1 . The components of J are given by

$$(13) \quad \begin{cases} J_{ij}^{(1)} = \sqrt{2ct} g_{ij} + b_1 p_i p_j, \\ J_{(2)}^{ij} = \frac{1}{\sqrt{2ct}} g^{ij} - \frac{b_1}{2\sqrt{ct}(\sqrt{c} + \sqrt{2tb_1})} g^{0i} g^{0j}. \end{cases}$$

4. A CLASS OF NATURAL HERMITIAN STRUCTURES ON T_0^*M

Consider the following symmetric M -tensor fields on T_0^*M , defined by the components

$$(14) \quad G_{ij}^{(1)} = c_1 g_{ij} + d_1 p_i p_j, \quad G_{(2)}^{ij} = c_2 g^{ij} + d_2 g^{0i} g^{0j},$$

where c_1, c_2, d_1, d_2 are smooth functions depending on the energy density $t \in (0, \infty)$.

Obviously, $G^{(1)}$ is of type $(0, 2)$ and $G_{(2)}$ is of type $(2, 0)$. We shall assume that the matrices defined by $G^{(1)}$ and $G_{(2)}$ are positive definite. This happens if and only if

$$(15) \quad c_1 > 0, \quad c_2 > 0, \quad c_1 + 2td_1 > 0, \quad c_2 + 2td_2 > 0 \quad \forall t > 0.$$

Then the following class of Riemannian metrics may be considered on T_0^*M

$$(16) \quad G = G_{ij}^{(1)} dq^i dq^j + G_{(2)}^{ij} Dp_i Dp_j,$$

where $Dp_i = dp_i - \Gamma_{ij}^0 dq^j$ is the absolute (covariant) differential of p_i with respect to the Levi Civita connection $\bar{\nabla}$ of g . Equivalently, we have

$$G\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right) = G_{ij}^{(1)}, \quad G\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = G_{(2)}^{ij}, \quad G\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right) = G\left(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}\right) = 0.$$

Remark that HT_0^*M , VT_0^*M are orthogonal to each other with respect to G , but the Riemannian metrics induced from G on HT_0^*M , VT_0^*M are not the same, so the considered metric G on T_0^*M is not a metric of Sasaki type. The $2n \times 2n$ -matrix associated to G , with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$ has two $n \times n$ -blocks on the first diagonal

$$G = \begin{pmatrix} G_{ij}^{(1)} & 0 \\ 0 & G_{(2)}^{ij} \end{pmatrix}.$$

The class of Riemannian metrics G is called a *class of natural lifts of diagonal type* of g .

Remark also that the system of 1-forms $(Dp_1, \dots, Dp_n, dq^1, \dots, dq^n)$ defines a local frame on $T^*T_0^*M$, dual to the local frame $(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n})$, on TT_0^*M over $\pi^{-1}(U)$ adapted to the direct sum decomposition (1).

We shall consider another two M -tensor fields $H_{(1)}$, $H^{(2)}$ on T_0^*M , defined by the components

$$\begin{cases} H_{(1)}^{jk} = \frac{1}{c_1} g^{jk} - \frac{d_1}{c_1(c_1 + 2td_1)} g^{0j} g^{0k}, \\ H_{jk}^{(2)} = \frac{1}{c_2} g_{jk} - \frac{d_2}{c_2(c_2 + 2td_2)} p_j p_k. \end{cases}$$

The components $H_{(1)}^{jk}$ define an M -tensor field of type $(2, 0)$ and the components $H_{jk}^{(2)}$ define an M -tensor field of type $(0, 2)$. Moreover, the

matrices associated to $H_{(1)}$, $H^{(2)}$ are the inverses of the matrices associated to $G^{(1)}$ and $G_{(2)}$, respectively. Hence we have

$$G_{ij}^{(1)} H_{(1)}^{jk} = \delta_i^k, \quad G_{(2)}^{ij} H_{jk}^{(2)} = \delta_k^i.$$

Now, we shall be interested in the conditions under which the class of the metrics G is Hermitian with respect to the class of the complex structures J , considered in the previous section, i.e.

$$G(JX, JY) = G(X, Y),$$

for all vector fields X, Y on T_0^*M .

Considering the coefficients of g_{ij}, g^{ij} in the conditions

$$(17) \quad \begin{cases} G(J \frac{\delta}{\delta q^i}, J \frac{\delta}{\delta q^j}) = G(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}), \\ G(J \frac{\partial}{\partial p_i}, J \frac{\partial}{\partial p_j}) = G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}), \end{cases}$$

we can express the parameters c_1, c_2 with the help of the parameters a_1, a_2 and a proportionality factor $\lambda = \lambda(t)$. Then

$$(18) \quad c_1 = \lambda a_1 = \sqrt{2ct}\lambda, \quad c_2 = \lambda a_2 = \frac{\lambda}{\sqrt{2ct}},$$

where the coefficients a_1, a_2 are given by (5) and (7). Since we made the assumption $c_1 > 0, c_2 > 0$, it follows $\lambda > 0$.

Next, considering the coefficients of $p_i p_j, g^{0i} g^{0j}$ in the relations (17), we can express the parameters $c_1 + 2td_1, c_2 + 2td_2$ with help of the parameters $a_1 + 2tb_1, a_2 + 2tb_2$ and a proportionality factor $\lambda + 2t\mu$

$$(19) \quad \begin{cases} c_1 + 2td_1 = (\lambda + 2t\mu)(a_1 + 2tb_1), \\ c_2 + 2td_2 = (\lambda + 2t\mu)(a_2 + 2tb_2). \end{cases}$$

Remark that $\lambda(t) + 2t\mu(t) > 0 \forall t > 0$. It is much more convenient to consider the proportionality factor in such a form in the expression of the parameters $c_1 + 2td_1, c_2 + 2td_2$. Using by the relations (5), (7), (11), (18) we can obtain easily from (19) the explicit expressions of the coefficients d_1, d_2

$$(20) \quad \begin{cases} d_1 = b_1 \lambda + \sqrt{2t}(\sqrt{c} + \sqrt{2tb_1})\mu, \\ d_2 = \frac{-b_1 \lambda + \sqrt{2ct}\mu}{2\sqrt{ct}(\sqrt{c} + \sqrt{2tb_1})}. \end{cases}$$

Hence we may state:

THEOREM 5. *Let J be the class of natural, complex structure of diagonal type on T_0^*M , given by (3) and (13). Let G be the class of the natural Riemannian metrics of diagonal type on T_0^*M , given by (14), (18), (20).*

*Then we obtain a class of Hermitian structures (G, J) on T_0^*M , depending on three essential parameters b_1 , λ and μ , which must fulfill the conditions*

$$(21) \quad b_1 > -\sqrt{\frac{c}{2t}}, \quad \lambda > 0, \quad \lambda + 2t\mu > 0 \quad \forall t > 0.$$

5. A CLASS OF KÄHLER STRUCTURES ON T_0^*M

Consider now the two-form ϕ defined by the class of Hermitian structures (G, J) on T_0^*M

$$\phi(X, Y) = G(X, JY),$$

for all vector fields X, Y on T_0^*M .

Using by the expression of ϕ and computing the values $\phi(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j})$, $\phi(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})$, $\phi(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j})$, we obtain.

PROPOSITION 6. *The expression of the 2-form ϕ in a local adapted frame $(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n})$ on T_0^*M , is given by*

$$\phi(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0, \quad \phi(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = 0, \quad \phi(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = \lambda \delta_j^i + \mu g^{0i} p_j,$$

or, equivalently

$$(22) \quad \phi = (\lambda \delta_j^i + \mu g^{0i} p_j) Dp_i \wedge dq^j.$$

THEOREM 7. *The class of Hermitian structures (G, J) on T_0^*M is Kähler if and only if*

$$\mu = \lambda'.$$

Proof. The expressions of $d\lambda$, $d\mu$, dg^{0i} and dDp_i are obtained in a straightforward way, by using the property $\dot{\nabla}_k g_{ij} = 0$ (hence $\dot{\nabla}_k g^{ij} = 0$)

$$d\lambda = \lambda' g^{0i} Dp_i, \quad d\mu = \mu' g^{0i} Dp_i, \quad dg^{0i} = g^{ik} Dp_k - g^{0h} \Gamma_{hk}^i dq^k,$$

$$dDp_i = -\frac{1}{2} R_{ikl}^0 dq^k \wedge dq^l + \Gamma_{ik}^l dq^k \wedge Dp_l.$$

Then we have

$$\begin{aligned} d\phi &= (d\lambda\delta_j^i + d\mu g^{0i}p_j + \mu dg^{0i}p_j + \mu g^{0i}dp_j) \wedge Dp_i \wedge dq^j + \\ &\quad + (\lambda\delta_j^i + \mu g^{0i}p_j)dDp_i \wedge dq^j. \end{aligned}$$

By replacing the expressions of $d\lambda, d\mu, dg^{0i}$ and $d\dot{\nabla}y^h$, then using, again, the property $\dot{\nabla}_k g_{ij} = 0$, doing some algebraic computations with the exterior products, then using the well known symmetry properties of g_{ij}, Γ_{ij}^h , and of the Riemann-Christoffel tensor field, as well as the Bianchi identities, it follows that

$$d\phi = \frac{1}{2}(\lambda' - \mu)g^{0h}Dp_h \wedge Dp_i \wedge dq^i.$$

Therefore we have $d\phi = 0$ if and only if $\mu = \lambda'$.

Remark. The class of natural Kähler structures of diagonal type defined by (G, J) on T_0^*M depends on two essential parameters b_1 and λ .

The paramaters b_1 and λ must fulfill the conditions

$$(23) \quad b_1 > -\sqrt{\frac{c}{2t}}, \quad \lambda > 0, \quad \lambda + 2t\lambda' > 0 \quad \forall t > 0.$$

The components of the class of Kähler metrics G on T_0^*M are given by

$$(24) \quad \begin{cases} G_{ij}^{(1)} = \sqrt{2ct}\lambda g_{ij} + [\sqrt{2ct}\lambda' + b_1(\lambda + 2t\lambda')]p_i p_j, \\ G_{(2)}^{ij} = \frac{\lambda}{\sqrt{2ct}}g^{ij} + \frac{-b_1\lambda + \sqrt{2ct}\lambda'}{2\sqrt{ct}(\sqrt{c} + \sqrt{2tb_1})}g^{0i}g^{0j}. \end{cases}$$

We obtain, too

$$(25) \quad \begin{cases} H_{(1)}^{jk} = \frac{1}{\sqrt{2ct}\lambda}g^{jk} - \frac{\sqrt{2ct}\lambda' + b_1(\lambda + 2t\lambda')}{2\sqrt{ct}\lambda(\sqrt{c} + \sqrt{2tb_1})(\lambda + 2t\lambda')}g^{0j}g^{0k}, \\ H_{jk}^{(2)} = \frac{\sqrt{2ct}}{\lambda}g_{jk} - \frac{-b_1\lambda + \sqrt{2ct}\lambda'}{\lambda(\lambda + 2t\lambda')}p_j p_k. \end{cases}$$

6. THE LEVI CIVITA CONNECTION OF METRIC G AND ITS CURVATURE TENSOR FIELD

The Levi Civita connection ∇ of the Riemannian manifold (T_0^*M, G) is determined by the conditions

$$\nabla G = 0, \quad T = 0,$$

where T is its torsion tensor field. The explicit expression of this connection is obtained from the formula

$$\begin{aligned} 2G(\nabla_X Y, Z) &= X(G(Y, Z)) + Y(G(X, Z)) - Z(G(X, Y)) + \\ &+ G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X); \quad \forall X, Y, Z \in \Gamma(T_0^*M). \end{aligned}$$

The final result can be stated as follows.

THEOREM 8. *The Levi Civita connection ∇ of G has the following expression in the local adapted frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$:*

$$(26) \quad \left\{ \begin{array}{ll} \nabla_{\frac{\partial}{\partial p_i}} \frac{\partial}{\partial p_j} = Q_h^{ij} \frac{\partial}{\partial p_h}, & \nabla_{\frac{\delta}{\delta q^i}} \frac{\partial}{\partial p_j} = -\Gamma_{ih}^j \frac{\partial}{\partial p_h} + P_i^{hj} \frac{\delta}{\delta q^h}, \\ \nabla_{\frac{\partial}{\partial p_i}} \frac{\delta}{\delta q^j} = P_j^{hi} \frac{\delta}{\delta q^h}, & \nabla_{\frac{\delta}{\delta q^i}} \frac{\delta}{\delta q^j} = \Gamma_{ij}^h \frac{\delta}{\delta q^h} + S_{hij} \frac{\partial}{\partial p_h}, \end{array} \right.$$

where $Q_h^{ij}, P_j^{hi}, S_{hij}$ are M -tensor fields on T_0^*M , defined by

$$(27) \quad \left\{ \begin{array}{l} Q_h^{ij} = \frac{1}{2} H_{hk}^{(2)} \left(\frac{\partial}{\partial p_i} G_{(2)}^{jk} + \frac{\partial}{\partial p_j} G_{(2)}^{ik} - \frac{\partial}{\partial p_k} G_{(2)}^{ij} \right), \\ P_j^{hi} = \frac{1}{2} H_{(1)}^{hk} \left(\frac{\partial}{\partial p_i} G_{jk}^{(1)} - G_{(2)}^{il} R_{ljk}^0 \right), \\ S_{hij} = -\frac{1}{2} H_{hk}^{(2)} \frac{\partial}{\partial p_k} G_{ij}^{(1)} + \frac{1}{2} R_{hij}^0. \end{array} \right.$$

Assuming that the base manifold (M, g) has positive constant sectional curvature c and the class of metrics G is Kähler, using by the relations (24),(25), one obtains

$$\left\{ \begin{aligned}
Q_h^{ij} &= \frac{\sqrt{c}-\sqrt{2}tb_1}{4\sqrt{ct}}g^{ij}p_h - \frac{\lambda-2t\lambda'}{4t\lambda}(\delta_h^i g^{0j} + \delta_h^j g^{0i}) + \\
&\quad \frac{\lambda(b_1^2\lambda+c\lambda')-\sqrt{2c}\lambda(b_1'\lambda-b_1\lambda')t^{1/2}+2(b_1'\lambda\lambda'-2c\lambda'^2+c\lambda\lambda'')t-2\sqrt{2c}\lambda'(\lambda b_1'+2b_1\lambda')t^{3/2}}{4\sqrt{ct}\lambda(\sqrt{c}+\sqrt{2}tb_1)(\lambda+2t\lambda')}g^{0i}g^{0j}p_h, \\
P_j^{hi} &= \frac{-\sqrt{c}\lambda+\sqrt{2}b_1\lambda t^{1/2}+2\sqrt{c}\lambda't+2\sqrt{2}b_1\lambda't^{3/2}}{4\sqrt{c}\lambda t}g^{hi}p_j + \\
&\quad \frac{1}{4t}\delta_j^i g^{0h} + \frac{\lambda+2t\lambda'}{4t\lambda}\delta_j^h g^{0i} + \frac{1}{2\sqrt{2c}\lambda t(\sqrt{2c}+2b_1\sqrt{t})(\lambda+2t\lambda')}[\lambda(-b_1^2\lambda+c\lambda')+ \\
&\quad \sqrt{2c}b_1'\lambda^2t^{1/2}-2(2b_1^2\lambda\lambda'+2c\lambda'^2-c\lambda\lambda'')t+ \\
&\quad 2\sqrt{2c}(b_1'\lambda\lambda'-3b_1\lambda'^2+b_1\lambda\lambda'')t^{3/2}-4b_1^2\lambda'^2t^2]g^{0h}g^{0i}p_j, \\
S_{hij} &= \frac{\sqrt{c}(\sqrt{c}+\sqrt{2}tb_1)}{2}g_{ij}p_h + \frac{\sqrt{c}(\sqrt{c}\lambda-\sqrt{2}b_1\lambda t^{1/2}-2\sqrt{c}\lambda't-2\sqrt{2}b_1\lambda't^{3/2})}{2\lambda}g_{hj}p_i - \\
&\quad \frac{\sqrt{c}(\sqrt{c}\lambda+\sqrt{2}b_1\lambda t^{1/2}+2\sqrt{c}\lambda't+2\sqrt{2}b_1\lambda't^{3/2})}{2\lambda}g_{hi}p_j - \\
&\quad \frac{1}{2\lambda(\lambda+2t\lambda')}[\lambda(2b_1^2\lambda+c\lambda')+\sqrt{2c}\lambda(\lambda b_1'+4\lambda'b_1)t^{1/2}+ \\
&\quad 2(\lambda^2b_1b_1'+5\lambda\lambda'b_1^2-2c\lambda'^2+c\lambda\lambda'')t+ \\
&\quad 2\sqrt{2c}(\lambda\lambda'b_1'-2\lambda'^2b_1+2\lambda\lambda''b_1)t^{3/2}+4\lambda b_1(\lambda'b_1'+\lambda''b_1)t^2]p_h p_i p_j.
\end{aligned} \right.$$

The curvature tensor field K of the connection ∇ is obtained from the well known formula

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \quad \forall X, Y, Z \in \Gamma(T_0^*M).$$

The components of curvature tensor field K with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \dots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n})$ are obtained easily:

$$(28) \left\{ \begin{aligned}
&K(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = QQQ_{ijk}^h \frac{\delta}{\delta q^h}, \quad K(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j})\frac{\partial}{\partial p_k} = QQP_{ijh}^k \frac{\partial}{\partial p_h}, \\
&K(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j})\frac{\delta}{\delta q^k} = PPQ_k^{ijh} \frac{\delta}{\delta q^h}, \quad K(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j})\frac{\partial}{\partial p_k} = PPP_h^{ijk} \frac{\partial}{\partial p_h}, \\
&K(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j})\frac{\delta}{\delta q^k} = PQQ_{jkh}^i \frac{\partial}{\partial p_h}, \quad K(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j})\frac{\partial}{\partial p_k} = PQP_j^{ikh} \frac{\delta}{\delta q^h},
\end{aligned} \right.$$

where

$$(29) \quad \left\{ \begin{array}{l} QQQ_{ijk}^h = R_{kij}^h - P_k^{hl} R_{lij}^0 + P_i^{hl} S_{ljk} - P_j^{hl} S_{lik}, \\ QQP_{ijh}^k = -R_{hij}^k - Q_h^{lk} R_{lij}^0 - P_i^{lk} S_{hjl} + P_j^{lk} S_{hil}, \\ PPQ_k^{ijh} = \frac{\partial}{\partial p_i} P_k^{hj} - \frac{\partial}{\partial p_j} P_k^{hi} + P_l^{hi} P_k^{lj} - P_l^{hj} P_k^{li}, \\ PPP_h^{ijk} = \frac{\partial}{\partial p_i} Q_h^{jk} - \frac{\partial}{\partial p_j} Q_h^{ik} + Q_l^{jk} Q_h^{il} - Q_l^{ik} Q_h^{jl}, \\ PQQ_{jkh}^i = \frac{\partial}{\partial p_i} S_{hjk} + Q_h^{il} S_{ljk} - P_k^{li} S_{hjl}, \\ PQP_j^{ikh} = \frac{\partial}{\partial p_i} P_j^{hk} + P_l^{hi} P_j^{lk} - P_j^{hl} Q_l^{ik}. \end{array} \right.$$

are M-tensor fields on T_0^*M .

Remark. The explicit expressions of these components are obtained after some quite long and hard computations, made by using the package RICCI.

From the local coordinates expression of the curvature tensor field K we obtain that the class of Kähler structures (G, J) on T_0^*M cannot have constant holomorphic sectional curvature.

The Ricci tensor field Ric of ∇ is defined by the formula:

$$Ric(Y, Z) = trace(X \longrightarrow K(X, Y)Z), \quad \forall X, Y, Z \in \Gamma(T_0^*M).$$

It follows

$$\left\{ \begin{array}{l} Ric(\frac{\delta}{\delta q^j}, \frac{\delta}{\delta q^k}) = RicQQ_{jk} = QQQ_{hjk}^h + PQQ_{jkh}^h, \\ Ric(\frac{\partial}{\partial p_j}, \frac{\partial}{\partial p_k}) = RicPP^{jk} = PPP_h^{hjk} - PQP_h^{jkh}, \\ Ric(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = Ric(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}) = 0. \end{array} \right.$$

Doing the necessary computations, we obtain the final expressions of the components of the Ricci tensor field of ∇

$$(30) \quad \left\{ \begin{array}{l} RicQQ_{jk} = \frac{a}{2\lambda(\lambda+2t\lambda')} g_{jk} + \frac{\alpha}{4t\lambda^2(\lambda+2t\lambda')^2} p_j p_k, \\ RicPP^{jk} = \frac{a}{4ct\lambda(\lambda+2t\lambda')} g^{jk} + \frac{\beta}{8\sqrt{ct^2\lambda^2(\sqrt{c}+\sqrt{2tb_1})(\lambda+2t\lambda')^2}} g^{0j} g^{0k}, \end{array} \right.$$

where the coefficient a is given by

$$a = c\lambda^2(n-2) - 8c\lambda\lambda't - 4c(n-1)\lambda'^2t^2 - 4c\lambda\lambda''t^2 - \\ \sqrt{2ct}[(n+1)\lambda^2 + 2\lambda\lambda'(2n+3)t + 4\lambda'^2(n-1)t^2 + 4\lambda\lambda''t^2]b_1 - \\ 2\sqrt{2c}\lambda t^{3/2}(\lambda + 2t\lambda')b_1'.$$

The expressions of α and β are too large to be presented in this paper.

7. A CLASS OF KÄHLER EINSTEIN STRUCTURES ON T_0^*M

In order to find out the conditions under which the class of Kähler structures (G, J) on T_0^*M is Einstein, we consider the relations

$$(31) \quad \begin{cases} RicQQ_{jk} = Ef G_{jk}^{(1)}, \\ RicPP^{jk} = Ef G_{(2)}^{jk}, \end{cases}$$

where the real number Ef is an Einstein factor.

Using by the relations (24), (30) and Lemma 1, one obtains the conditions

$$Ef = \frac{a}{2\lambda^2\sqrt{2ct}(\lambda + 2t\lambda')} = \frac{\alpha}{4t\lambda^2(\lambda + 2t\lambda')^2[\sqrt{2ct}\lambda' + (\lambda + 2t\lambda')b_1]} = \\ = \frac{\beta}{4t\lambda^2(\lambda + 2t\lambda')^2(\sqrt{2ct}\lambda' - \lambda b_1)}.$$

The condition

$$Ef = \frac{a}{2\lambda^2\sqrt{2ct}(\lambda + 2t\lambda')}$$

is equivalent with the first order differential equation

$$2\sqrt{2c}\lambda t^{3/2}(\lambda + 2t\lambda')b_1' = -\sqrt{2ct}[(n+1)\lambda^2 + 2\lambda\lambda'(2n+3)t + 4\lambda'^2(n-1)t^2 + 4\lambda\lambda''t^2]b_1 + \\ c\lambda^2(n-2) - 8c\lambda\lambda't - 4c(n-1)\lambda'^2t^2 - 4c\lambda\lambda''t^2 - Ef 2\sqrt{2ct}\lambda^2(\lambda + 2t\lambda').$$

The solution of this equation is

$$(32) \quad b_1(t) = \frac{t^{-\frac{n+1}{2}}\lambda^{1-n}}{4(\lambda + 2t\lambda')}(4C - \int_1^t \theta(s)ds), \quad t > 0, \quad C \in \mathbf{R},$$

where

$$\theta(s) = s^{\frac{n-2}{2}} \lambda^{n-2} \{4 E f \sqrt{s} \lambda^2 (\lambda + 2s\lambda') + \sqrt{2c} [-(n-2)\lambda^2 + 4(n-1)s^2\lambda'^2 + 4s\lambda(2\lambda' + s\lambda'')]\}.$$

If the function b_1 is given by (32) then the relations

$$Ef = \frac{\alpha}{4t\lambda^2(\lambda + 2t\lambda')^2[\sqrt{2ct}\lambda' + (\lambda + 2t\lambda')b_1]} = \frac{\beta}{4t\lambda^2(\lambda + 2t\lambda')^2(\sqrt{2ct}\lambda' - \lambda b_1)}$$

are fulfilled.

Now we may state our main result.

THEOREM 9. *Assume that the Riemannian manifold (M, g) has positive constant sectional curvature c . Let J be the class of natural, complex structure of diagonal type on T_0^*M , given by (3) and (13). Let G be the class of the natural Riemannian metrics of diagonal type on T_0^*M , given by (14) and (24).*

*If b_1 is given by (32) then (G, J) is a class of Kähler Einstein structures on T_0^*M , depending on one essential parameter λ .*

The function b_1 and the parameter λ must fulfill the conditions (23):

$$b_1 > -\sqrt{\frac{c}{2t}}, \quad \lambda > 0, \quad \lambda + 2t\lambda' > 0 \quad \forall t > 0.$$

Remark. After some long and hard computations we have obtained that the class of Kähler Einstein structures (G, J) on T_0^*M is not locally symmetric.

Example. The case where $\lambda = 1$ is presented in [10].

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